LIE TRANSFORM CONSTRUCTION OF EXPONENTIAL NORMAL FORM OBSERVERS

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Abstract: Motivated by a class of nonlinear regulator problems associated with systems that experience actuator failure, this work discusses the construction of nonlinear observers in normal form using a systematic and elegant method of transformation, namely Lie Transforms. It is shown how the transformation, which is the solution of a Homological Equation, and the observer gain matrix, which is a series of matrix homogenous polynomials, can be sequentially computed. An illustrative example is given and experience with using these observers is discussed.

Keywords: Nonlinear, Observers, Exponential, Series, Resonance

1. INTRODUCTION

In control system design there is often a need to estimate the state variables. The states to be estimated may include both process and disturbance states. Recently, we have been interested in reconfigurable control systems for aircraft with jammed actuators, e.g. (Bajpai *et al.*, 2002). We formulate the damaged system control design problem as a nonlinear regulator problem in which we confront the problem of designing a nonlinear composite observer.

Nonlinear observer design has been an area of active research for some time. Krener and Isidori (Krener and Isidori, 1983) and Krener and Respondek (Krener and Respondek, 1985) considered the problem of synthesis of observers yielding error dynamics that are linear in transformed coordinates. However, the necessary conditions are quite restrictive.

Kazantis and Kravaris (Kazantzis and Kravaris, 1998) proposed the construction of 'normal form' observers with linear dynamics based on Lyapunov's auxiliary theorem. The necessary conditions still pose undesirable restrictions because of the requirement that the eigenvalues of the linearized plant lie in the Poincarè domain. Krener and Xiao (Krener and Xiao, 2001) extended the observer design method to the Siegel domain. The result can be applied to any real analytic zero-input linearly observable system. A method for constructing an observer with approximately linear error dynamics by polynomial approximation of the solution to the partial differential equation was outlined.

For large systems a more efficient approach is required. Normal form observers can be constructed systematically and elegantly using the Lie Transformation. That is the subject of this paper.

In Section 2 we describe some general properties of exponential observers for autonomous nonlinear dynamics and in Section 3 we summarize the normal form approach for designing observers. Section 4 gives the Lie transform method for constructing the required transformations. These calculations have been implemented in Mathemat-

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ica. Finally, we give an example in Section 5 and conclusions in Section 6.

2. EXPONENTIAL OBSERVERS

Consider the system

$$\begin{aligned}\dot{x} &= f(x)\\ y &= h(x)\end{aligned} \tag{1}$$

where $x \in R^n$, $y \in R^p$, and f, h are smooth vector fields with $f(0) = 0, h(0) = 0$. This implies that the functions possess a formal Taylor series representation. The system (1) is assumed to be locally observable at $x = 0$.

2.1 Basic Definitions and Properties

Definition 2.1. The system (1) is said to be exponentially detectable if there exists a function $\gamma(\xi, y)$ defined on a neighborhood of $(\xi, y) = (0, 0)$ that satisfies:

1. $\gamma(0, 0) = 0$
2. $\gamma(\xi, h(\xi)) = f(\xi)$
3. $\xi = 0$ is an exponentially stable equilibrium point of $\dot{\xi} = \gamma(\xi, 0)$

A system whose linearization is detectable is exponentially detectable. Exponential detectability implies that the system

$$\dot{\widehat{x}} = \gamma(\widehat{x}, y) \tag{2}$$

is a local observer for (1) in the sense that $\|x(t) - \widehat{x}(t)\| \rightarrow 0$ as $t \rightarrow \infty$ provided $x(t)$ remains sufficiently close to to the origin, (Kwatny and Blankenship, 2000).

Lemma 2.2. The system

$$\dot{z} = \eta(z, y), \quad z \in R^n \tag{3}$$

$$\widehat{x}(t) = T(z(t)) \tag{4}$$

is a full state exponential observer for (1) if $\eta(0, 0) = 0, z = 0$ is an exponentially stable equilibrium point of (3) with $y = 0$, and $T : R^n \rightarrow R^n$ is a locally smooth, invertible mapping such that $x(t) = T(z(t))$ for each $x(0) = T(z(0))$ on a neighborhood of $x = 0$.

Proof: Since

$$\dot{x} = \left[\frac{\partial T(z)}{\partial z}\eta(z, y)\right]_{z \rightarrow S(x), y \rightarrow h(x)} = f(x)$$

where $z(t) = S(x(t))$ is the inverse transformation of $x(t) = T(z(t))$. It follows that the function

$$\gamma(x, y) := \left[\frac{\partial T(z)}{\partial z}\eta(z, y)\right]_{z \rightarrow S(x)} \tag{5}$$

satisfies the three conditions of Definition 2.1. ■

2.2 Observers with Linear Error Dynamics

We seek an exponential observer of the type (3), (4) with linear dynamics, i.e.,

$$\dot{z} = Az - L_0 y, \quad z \in R^n \tag{6}$$

$$\widehat{x}(t) = T(z(t)) \tag{7}$$

Remark 2.3. If such an observer exists, a simple calculation (see (Kazantzis and Kravaris, 1998)) shows that the error dynamics are linear when expressed in the z-coordinates, specifically

$$\frac{d}{dt}(S(x) - S(\widehat{x})) = A(S(x) - S(\widehat{x}))$$

We attempt to build an observer of the form (6) by direct construction. The formal Taylor series of (1) at $x = 0$ is

$$\dot{x} = Fx + F_2(x) + \cdots + F_r(x) + O(|x|^{r+1})$$

$$y = Hx + H_2(x) + \cdots + H_r(x) + O(|x|^{r+1})$$

where the components $F_k(x), H_k(x)$ are vector homogenous polynomials in the elements of x of degree k. The dynamics of (1) can be recast as

$$\dot{x} = f(x) + L(x)h(x) - L(x)y$$

where the matrix $L(x)$ is the observer gain that has to be designed. We specify $L(x)$ in the form

$$L(x) = [I + L_1(x) + L_2(x) + \cdots]L_0$$

where the $L_k(x)$ are matrix homogenous polynomials of degree k in the elements of x. Express $L_k(x)$ $k = 1, 2, \ldots$ as matrix homogenous polynomials with unknown coefficients. Using the expansions of f, h and L

$$\dot{x} = Ax + \sum_{k \geq 1}\frac{f_k(x)}{k!} - \left[I + \sum_{k \geq 1} L_k(x)\right] L_0 y \tag{8}$$

where

$$A = F + L_0 H$$

$$\frac{f_k(x)}{k!} = F_{k+1}(x) + L_0 H_{k+1}(x) + \cdots$$
$$\cdots + L_k(x) L_0 H x$$

The idea is to find a transformation $x = T(z)$ of the form

$$x = T_0 z + T_1(z) + T_2(z) + \cdots$$

such that nonlinear terms are eliminated from the transformed equations and result in an equation of the form (6).

3. REDUCTION TO NORMAL FORM

Definition 3.4. An n-tuple $(\lambda_1, \ldots, \lambda_n)$ of eigenvalues belongs to the Poincarè domain if the convex hull of the n points $(\lambda_1, \ldots, \lambda_n)$ in the complex plane does not contain zero. An n-tuple of eigenvalues belong to the Siegel domain if zero lies in the convex hull of$(\lambda_1, \ldots, \lambda_n)$.

Definition 3.5. The n-tuple $(\lambda_1, \ldots, \lambda_n)$ of eigenvalues of A is said to be resonant if there exists a relation among the eigenvalues of the form

$$\lambda_s = m_1\lambda_1 + \cdots + m_n\lambda_n, \quad s \in \{1, \ldots, n\}$$
$$m_k \geq 0, \quad \sum m_k \geq 2$$

Proposition 3.6. (Poincarè-Siegel Theorem) Suppose the eigenvalues of A are nonresonant and the vector field $v(x)$ is given by the formal power series

$$v(x) = Ax + v_2(x) + v_3(x) + \cdots$$

Then $v(x)$ is reducible to the linear vector field

$$w(z) = Az$$

by a near identity, formal power series change of variables.

If, in addition, the eigenvalues of A belong to the Poincarè domain and the vector field is analytic, then the transformation is analytic (the series converges).

Proof: (Arnold, 1983), Chapter 5. ■

Remark 3.7. Since (F, H) is an observable pair there exists a matrix L_0 (indeed many) such that the matrix $A = F + L_0H$ is asymptotically stable and its eigenvalues are nonresonant. Then, the above theorem can be used to establish the existence of a near identity transformation of (8) into (6). Furthermore, the analyticity requirement that the eigenvalues of A belong to the Poincarè domain can be eliminated if a stronger form of nonresonance is assumed.

Definition 3.8. A complex number λ is said to be of type (C, ν) with respect to the spectrum of $F = \sigma(F) = (\alpha_1, \ldots \alpha_n)$ if for any vector $m = (m_1, \ldots, m_n)$ of nonnegative numbers, $|m| = \sum m_i > 0$, we have

$$|\lambda - m \cdot \alpha| \geq \frac{C}{|m|^\nu}$$

where $C > 0, \nu > 0$ are constants.

Proposition 3.9. Suppose all the eigenvalues of A are of type (C, ν) with respect to $\sigma(F)$. Then the transformation of Proposition 3.6 is analytic in some neighborhood of the origin.

Proof: (Arnold, 1983) ■

Proposition 3.10. For each L_0 that renders $A = F + L_0H$ asymptotically stable and nonresonant in the sense that

$$\lambda_i \neq m_1\alpha_1 + \cdots + m_n\alpha_n$$
$$i = 1, \ldots, n, m_k \geq 0, \sum m_k \geq 2$$

where λ_i, α_i are respectively, the eigenvalues of A and$F$, there exists a formal power series change of variables $x = T(z)$ such that

$$\dot{z} = Az - L_0 y \tag{9}$$
$$\widehat{x}(t) = T(z(t)) \tag{10}$$

is an exponential observer for (1).

Proof: This follows from direct application of the Poincarè-Siegel Theorem. See also (Krener and Xiao, 2001). ■

Remark 3.11. Notice that observability of (F, H) is not required. Detectability is sufficient provided that nonresonance condition is satisfied.

Remark 3.12. The observer can be implemented as shown in Equations (9) and (10) or as an 'identity' observer:

$$\dot{\widehat{x}} = f(\widehat{x}) + L(\widehat{x})(y - h(\widehat{x})) \tag{11}$$

Notice that because $L(x)$ is generated as a power series, we have a natural notion of observer 'order' associated with the degree of the terms retained in the expansion. The zeroth order observer, corresponding to $L(x) = L_0$ is the frequently used 'constant gain' observer.

Remark 3.13. As suggested in (Krener and Xiao, 2001) it may be advantageous to seek nonlinear output injection in the transformed system, i.e.,

$$\dot{z} = Az - \ell(y) \tag{12}$$

where ℓ is smooth and of the form

$$\ell(y) = L_0 y + h.o.t \tag{13}$$

This injects additional degrees of freedom (the coefficients of ℓ) that may be used to enlarge the domain of convergence of the transformation.

4. COMPUTATION VIA LIE TRANSFORMS

Scale the state variables x according to $x \to \varepsilon x$, where ε is a scalar parameter, so that (8) becomes

$$\begin{aligned} \dot{x} = Ax &+ \sum_{k\geq 1} \frac{f_k(x)}{k!}\varepsilon^k \\ &- \frac{1}{\varepsilon}\left[I + \sum_{k\geq 1} L_k(x)\varepsilon^k \right] L_0 y \end{aligned} \tag{14}$$

Now let $U(\bar{T}, \varepsilon)$ be a given 'generating function' and suppose the transformation $x = \bar{T}(z, \varepsilon)$ is defined as the solution of the equation

$$\frac{\partial \bar{T}}{\partial \varepsilon} = U(\bar{T}, \varepsilon), \quad \bar{T}(z, 0) = z$$

In new coordinates the system equations are

$$\begin{aligned} \dot{z} = Az &+ \sum_{k\geq 1} \frac{g_k(z)}{k!}\varepsilon^k \\ &- \frac{1}{\varepsilon}\left[I + \sum_{k\geq 1} L_k(\bar{T}(z,\varepsilon))\varepsilon^k \right] L_0 y \end{aligned}$$

where the components of g_k are homogeneous polynomials in z of degree$k + 1$.

Proposition 4.14. Suppose that U admits series expansion

$$U(u, \varepsilon) = \sum_{m=0}^{\infty} U_m(x)\varepsilon^m/m!$$

Define the sequence

$$f_i^{(m)}(x), \;\; i, m = 0, 1, 2, \ldots,$$

by the recursive relations

$$\begin{aligned} f_i^m = f_{i+1}^{(m-1)} &- \sum_{0\leq j\leq i} C_j^i ad_{f_{i-j}^{(m-1)}} U_j \\ i = 0, 1, 2, \ldots & \qquad m = 1, 2, \ldots \end{aligned} \tag{15}$$

$$f_i^{(0)} = f_i, i = 0, 1, 2, \ldots \tag{16}$$

where $C_j^i = i!/(j!(i-j)!$ is the binomial coefficient. Then

$$g_m = f_0^{(m)}, \quad m = 0, 1, 2, \ldots \tag{17}$$

Proof: (Chow and Hale, 1982), Chapter 12 ■

Remark 4.15. The computations can be organized according to the following triangle.

$$\begin{array}{lllll} f_0^{(0)} & & & & \\ f_1^{(0)} & f_0^{(1)} & & & \\ f_2^{(0)} & f_1^{(1)} & f_0^{(2)} & & \\ f_3^{(0)} & f_2^{(1)} & f_1^{(2)} & f_0^{(3)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

The i^{th} element of the m^{th} column of this triangle can be computed by knowing only the first $i + 2$ elements of the $(m-1)^{th}$ column. The f_i's are along the first column and the g_i 's are along the diagonal.

Our objective is to determine the generator $U(x, \varepsilon)$, from which we can obtain the transformation $u(z, \varepsilon)$, that takes

$$f(x, \varepsilon) = Ax + \sum_{k\geq 1} \frac{f_k(x)}{k!}\varepsilon^k \tag{18}$$

into

$$g(z, \varepsilon) = \sum_{m=0}^{\infty} g_m(z)\varepsilon^m/m! = Az \tag{19}$$

In particular, we require

$$g_0(z) = Az, \quad g_m(z) = 0, \; m \geq 1$$

Proposition 4.16. The generator components U_i for the transformation that takes the vector field (18) into (19) are given by

$$\begin{aligned} ad_{Ax}U_i \;&=\; f_{i+1} - \sum_{0\leq j\leq i-1} C_j^i ad_{f_{i-j}} U_j \\ i \;&=\; 1, 2, \ldots \end{aligned} \tag{20}$$

$$ad_{Ax}U_0 = f_1 \tag{21}$$

Proof: Let us compute the generator components U_j from (15) through (17), organizing the calculations in accordance with the table.

$$\begin{aligned}
&f_0^{(0)} = Ax \\
&f_1^{(0)} = f_1, \;\; f_0^{(1)} = f_1^{(0)} - ad_{f_0^{(0)}} U_0 \\
&\quad \Rightarrow ad_{Ax}U_0 = f_1 \\
&f_2^{(0)} = f_2, \\
&\quad f_1^{(1)} = f_2^{(0)} - C_0^1 ad_{f_1^{(0)}} U_0 - C_1^1 ad_{f_0^{(0)}} U_1, \\
&\quad f_0^{(2)} = f_1^{(1)} = 0 \\
&\quad \Rightarrow ad_{Ax}U_1 = f_2 - C_0^1 ad_{f_1} U_0 \\
&f_3^{(0)} = f_3, \\
&\quad f_2^{(1)} = f_3^{(0)} - C_0^2 ad_{f_2^{(0)}} U_0 \\
&\qquad - C_1^2 ad_{f_1^{(0)}} U_1 - C_2^2 ad_{f_0^{(0)}} U_2, \\
&\quad f_1^{(2)} = f_2^{(1)} - C_0^1 ad_{f_1^{(1)}} U_0 - C_1^1 ad_{f_0^{(1)}} U_1, \\
&\quad f_0^{(3)} = f_1^{(2)} = 0 \\
&\quad \Rightarrow ad_{Ax}U_2 = f_3 - C_0^2 ad_{f_2} U_0 - C_1^2 ad_{f_1} U_1
\end{aligned}$$

In general, we obtain (20). ■

To solve (20), (21) we need the following Lemma.

Lemma 4.17. Consider the operator ad_{Ax} that takes vector fields whose components are homogenous polynomials of degree m into the same linear vector space. If the eigenvalues of A are $\{\lambda_1, \dots, \lambda_n\}$ then the eigenvalues of ad_{Ax} are given by

$$\left\{\sum_{i=1}^{n} m_i\lambda_i - \lambda_j\right\}$$

$$\sum_{i=1}^{n} m_i = m \quad j = 1, \dots, n$$

Moreover, if A is diagonal then the operator ad_{Ax} is also diagonal on the space of homogenous vector-valued polynomials.

Proof: (Arnold, 1983), Chapter 12. ■

Remark 4.18. To solve the homological equation (20), (21) using the above Lemma we will first have to transform the system so that A is diagonal.

Remark 4.19. L_0 should be chosen so that none of the eigenvalues of ad_{Ax} are zero to ensure that the Homological equation has a unique solution.

Now that the generator U is known, we wish to determine the transformation $\bar{T}(z, \varepsilon)$ that satisfies (9).

Proposition 4.20. Define the sequence $p_i^{(m)}, i, m = 0, 1, 2, \dots$ by the recursive relations

$$p_i^{(m)} = p_{i+1}^{(m-1)} + \sum_{0 \le j \le i} C_j^i L_{p_{i-j}^{(m-1)}} U_j \qquad i = 0, 1, 2, \dots, \quad m = 1, 2, \dots \tag{22}$$

If $p_i^{(0)} = U_i$, $i = 0, 1, 2, \dots$, then $\bar{T}_{m+1} = p_0^{(m)}$.

Proof: (Chow and Hale, 1982), Chapter 12. ■

Remark 4.21. Notice that the computations of $p_0^{(m)}$ as given by (18) proceed along the same triangular structure as $f_0^{(m)}$. See Remark 13.

$$\begin{array}{lllll} p_0^{(0)} & & & & \\ p_1^{(0)} & p_0^{(1)} & & & \\ p_2^{(0)} & p_1^{(1)} & p_0^{(2)} & & \\ p_3^{(0)} & p_2^{(1)} & p_1^{(2)} & p_0^{(3)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Remark 4.22. Notice that in view of (8) f_m is a homogenous polynomial of degree $m+1$. Hence (15) implies that U_m is a homogenous polynomial of degree $m+1$, and (18) implies that $\bar{T}_m$ is a homogenous polynomial of degree $m+1$.

The k^{th} order transformation $x = \bar{T}_k(z) = z + \widetilde{T}_k$, where $\widetilde{T}_k = T_1/1! + \cdots + T_k/k!$, thus obtained is in terms of the unknown polynomial coefficients of $L(x)$. The transformation transforms (7) into

$$\left\{I + \frac{\partial \widetilde{T}_k}{\partial z}\right\} \dot{z} = A(z + \widetilde{T}_k) + \sum_{j \ge 1} \frac{f_j(z + \widetilde{T}_k)}{j!} - \left[I + \sum_{j \ge 1} L_j(z + \widetilde{T}_k)\right] L_0 y$$

Retaining terms of order k we write

$$\begin{aligned} \left\{I + \frac{\partial \widetilde{T}_k}{\partial z}\right\} (Az + L_0 y) &= A(z + \widetilde{T}_k) \\ &+ \sum_{j=1}^{k} \frac{f_j(z + \widetilde{T}_k)}{j!} \\ &- \left[I + \sum_{j=1}^{k-1} L_j(z + \widetilde{T}_k)\right] L_0 y \\ &+ O\left(|z|^{k+1}\right) \end{aligned}$$

The transformation T_k is constructed so that $f(x) \to Az + O(|z|^{k+1})$ or, equivalently, $ad_{Az}\widetilde{T}_k = \sum_{j=1}^{k} f_j/j!$. Thus, the unknown polynomial coefficients of $L(x)$ are determined from

$$\sum_{j=1}^{k-1} L_j = \frac{\partial \widetilde{T}_k}{\partial z}$$

Remark 4.23. The observer gain matrix $L(x) = \partial \bar{T} / \partial z \big|_{z \to \bar{T}^{-1}(x)}$ is real even if the eigenvalues of A are complex.

5. EXAMPLE

Numerous low order examples have been solved, including those in (Kazantzis and Kravaris, 1998), (Krener and Xiao, 2001) and (Krener and Xiao, 2002), in order to verify the computations. Here is the Van der Pol system from (Krener and Xiao, 2001).

$$\dot{x}_1 = x_2, \; \dot{x}_2 = \left(1 - x_1^2\right) x_2 - x_1$$

We specify $L_0 = [-2, -4]^T$ which results in the eigenvalues $-0.5 \pm j\,1.65831$. The transformation $T(z)$ and its inverse are:

$$T(z) = \begin{bmatrix} z_1 - 0.101852\, z_1{}^3 + 0.017789\, z_1{}^5 + 0.0277778\, z_1{}^2\, z_2 - \\ 0.0132345\, z_1{}^4\, z_2 + 0.0277778\, z_1\, z_2{}^2 - \\ 0.00547641\, z_1{}^3\, z_2{}^2 - 0.0185185\, z_2{}^3 + \\ 0.0102185\, z_1{}^2\, z_2{}^3 - 0.00366901\, z_1\, z_2{}^4 + \\ 0.00030489\, z_2{}^5 \\ \\ -0.231481\, z_1{}^3 + 0.0997385\, z_1{}^5 + \\ z_2 - 0.277778\, z_1{}^2\, z_2 + 0.0123547\, z_1{}^4\, z_2 + \\ 0.222222\, z_1\, z_2{}^2 - 0.0931538\, z_1{}^3\, z_2{}^2 - \\ 0.0648148\, z_2{}^3 + 0.0632284\, z_1{}^2\, z_2{}^3 - \\ 0.0139039\, z_1\, z_2{}^4 + 0.00155164\, z_2{}^5 \end{bmatrix}$$

$$T^{-1}(x) = \begin{bmatrix} x_1 + 0.101852\, x_1{}^3 + 0.00690236\, x_1{}^5 \\ -0.0277778\, x_1{}^2\, x_2 - 0.0214877\, x_1{}^4\, x_2 - \\ 0.0277778\, x_1\, x_2{}^2 - 0.000696427\, x_1{}^3\, x_2{}^2 + \\ 0.0185185\, x_2{}^3 + 0.0237321\, x_1{}^2\, x_2{}^3 - \\ 0.0125347\, x_1\, x_2{}^4 + 0.00278153\, x_2{}^5 \\ \\ 0.231481\, x_1{}^3 + 0.0352924\, x_1{}^5 + \\ x_2 + 0.277778\, x_1{}^2\, x_2 - 0.00078059\, x_1{}^4\, x_2 - \\ 0.222222\, x_1\, x_2{}^2 - 0.104377\, x_1{}^3\, x_2{}^2 + \\ 0.0648148\, x_2{}^3 + 0.111154\, x_1{}^2\, x_2{}^3 - \\ 0.0416517\, x_1\, x_2{}^4 + 0.00693601\, x_2{}^5 \end{bmatrix}$$

Typical responses are shown in Figure 1.

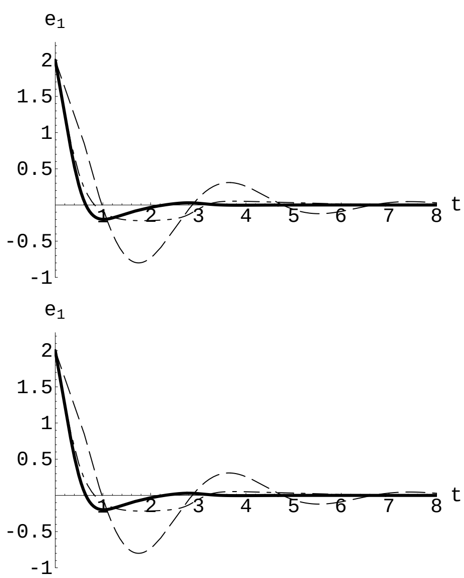

Fig. 1. The constant gain observer error response (solid) is compared with the 4^{th}-order normal form (identity) observer (long-short dashed) and the nonlinear output injection observer (long dashed) for initial conditions: $x_1(0) = 2,\ x_2(0) = .5,\ \widehat{x}_1(0) = 0,\ \widehat{x}_2(0) = 0$.

6. CONCLUSIONS

We have shown how to construct exponential observers with linear dynamics using Lie Transforms for systems that are zero-input observable and have a formal Taylor series representation. The nonresonance conditions can be easily met by proper choice of the eigenvalues of the observer matrix A.

The normal form computations based on Lie transforms have been implemented in Mathematica. For comparison, we have also implemented the 'brute force' method based on comparing coefficients of a series expansion. Observers for several simple examples have been computed. Even in these examples the benefits of the Lie transform method is noticeable.

Our experience indicates that observer implementation in 'identity' form is by far the most reliable. The most surprising result is the effectiveness of the constant gain observer. Any improvement by the higher order observers is marginal at best and often degrades if the series does not converge rapidly. This point highlights the advantage of nonlinear output injection which can be used to improve convergence. These comments are based on our experience with relatively simple problems.

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